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ON A LATTICE POINT PROBLEM OF L. MOSER II

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In this paper we complete the proof of the following conjecture of L. Moser: Any convex region of area n can be placed on the plane so as to cover $\geqq n+f(n)$ lattice points, where $f(n)\to\infty$.

0. Introduction

We recall the main result from Part I of this paper (see Section 1 in [1]).

Theorem 1.1. *There is a universal function $f(x), f(x)\geqq x^{1/9}$ for $x\geqq c_0$ (where c_0 is an "ineffective" absolute constant) such that any convex region A of area x can be placed on the plane so as to cover at least $x+f(x)$ (or at most $x-f(x)$) lattice points.*

For any compact and convex region B on the plane, let $d(B)$ and $r(B)$ denote the diameter of B and the radius of the largest inscribed circle of B, respectively. Let μ denote the two-dimensional Lebesgue measure (i.e. the usual area).

For any bounded set $S\subset\mathbf{R}^2$ and $\mathbf{x}\in\mathbf{R}^2$, let

$$g(S,\mathbf{x}) = \operatorname{card}(S+\mathbf{x})\cap\mathbf{Z}^2,$$

i.e., the number of lattice points covered by the translate $S+\mathbf{x}$ of S. For any positive real number $\varepsilon\in(0, 1/2]$, let

$$g(S,\mathbf{x},\varepsilon) = \frac{1}{4\varepsilon^2}\int\limits_{[-\varepsilon,\varepsilon)^2} g(S,\mathbf{x}+\mathbf{y})\,d\mathbf{y}.$$

Note that $g(S, \mathbf{x}, 1/2)=\mu(S)$ if S is Lebesgue-measurable and $\lim\limits_{\varepsilon\to 0} g(S,\mathbf{x},\varepsilon)=g(S,\mathbf{x})$ if there is no lattice point on the boundary of $S+\mathbf{x}$.

Given any angle $\tau\in[0, 2\pi)$, let τS denote the rotated image of $S\subset\mathbf{R}^2$. Let $\mathcal{U}^2=[0, 1)^2$. In Section 2 of [1], we have derived Theorem 1.1 from the following estimate.

Theorem 2.1. *There exist an "ineffective" absolute constant c_0 and an "effective" absolute constant $c_1>0$ such that for any convex region A with $\mu(A)\geqq c_0$ and*

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$r(A) \geqq 1/9$, *we have with* $\varepsilon_0 = (d(A))^{-1/100}$,

$$\frac{1}{2\pi} \int_0^{2\pi} \Big(\int_{\mathcal{U}^2} (g(\tau A, \mathbf{y}, \varepsilon_0) - \mu(A))^2 \, d\mathbf{y} \Big) \, d\tau \geqq c_1 \cdot (d(A))^{97/100}.$$

In Sections 3—4, we have derived Theorem 2.1, using Fourier Analysis, from the following two lemmas.

For any $\beta \in [0, 2\pi)$, write $\mathbf{e}(\beta) = (\cos\beta, \sin\beta)$. Let $h_{A+\mathbf{v}}(\beta, y)$ be the Euclidean length of the chord

$$\{\mathbf{y} \in A + \mathbf{v}\colon \ \mathbf{y} \cdot \mathbf{e}(\beta) = y\}.$$

We say that $h_{A+\mathbf{v}}(\beta, y)$, $\beta \in [0, 2\pi)$, $y \in \mathbf{R}$ is the *chord function* of $A + \mathbf{v}$.

Let

$$M^+_{\beta,\mathbf{v}} = \sup \{x \in \mathbf{R}\colon \ h_{A+\mathbf{v}}(\beta, x) > 0\}$$

and

$$M^-_{\beta,\mathbf{v}} = \inf \{x \in \mathbf{R}\colon \ h_{A+\mathbf{v}}(\beta, x) > 0\}.$$

Clearly $D_\beta = (M^+_{\beta,\mathbf{v}} - M^-_{\beta,\mathbf{v}})$ is the length of the projection of A onto a straight line parallel to the unit vector $\mathbf{e}(\beta)$.

Let $\varepsilon = \varepsilon_0 = (d(A))^{-1/100}$. Let $\eta \in (0, 1/100)$. Let $\{\xi\}$ denote the *fractional part* of the real number ξ, i.e., $\xi = [\xi] + \{\xi\}$.

We shall denote the distance from the real number ξ to the nearest integer by $\|\xi\|$.

For any $\beta \in [0, 2\pi)$, write

$$V(\beta) = V_\eta(\beta) = \Big\{\mathbf{v} \in \mathbf{R}^2\colon \ |\mathbf{v}| \leqq 1 \quad \text{and one can find positive integers}$$

$$k = k(\beta, \mathbf{v}), \quad l = l(\beta, \mathbf{v}) \quad \text{such that}$$

$$\frac{1}{10\varepsilon_0} \leqq (k^2 + l^2)^{1/2} \leqq \frac{1}{5\varepsilon_0}, \quad \text{and furthermore,}$$

$$\|(k^2 + l^2)^{1/2} \cdot M^-_{\beta,\mathbf{v}}\| \leqq 3\eta \quad \text{and} \quad \eta \leqq \{(k^2 + l^2)^{1/2} M^+_{\beta,\mathbf{v}}\} \leqq 2\eta \Big\},$$

where $\{\mathbf{v}\colon |\mathbf{v}| \leqq 1\} = \{\mathbf{v} = (v_1, v_2)\colon \ v_1^2 + v_2^2 \leqq 1\}$ is the unit disc.

We are now able to formulate the two lemmas

Lemma 4.1. *If* $1/100 \geqq \eta \geqq 2 \cdot (d(A))^{-10^{-5}}$ *and* $\mu(A)$ *is larger than an "ineffective" absolute constant, then* $\mu(V(\beta)) = \mu(V_\eta(\beta)) \geqq \eta$ *uniformly for all* $\beta \in [0, 2\pi)$.

The second one is a purely geometric lemma.

Given a convex region B, an angle $\beta \in [0, 2\pi)$ and a real number $y \geqq 0$, write

$$f_B(\beta, y) = h_{B+\mathbf{v}}(\beta, M^-_{\beta,\mathbf{v}} + y) \tag{0.1}$$

where

$$M^-_{\beta,\mathbf{v}} = M^-_{\beta,\mathbf{v}}(B) = \inf \{x \in \mathbf{R}\colon \ h_{B+\mathbf{v}}(\beta, x) > 0\}.$$

Observe that the right-hand side term in (0.1) is independent of the value of $\mathbf{v} \in \mathbf{R}^2$.

Lemma 4.2. *There are ("effective") positive absolute constants c_9, c_{10} and c_{11} such that for any convex region B with $r(B) \geqq c_9$,*

$$c_{10} \cdot d(B) \geqq \int_0^{2\pi} (f_B(\beta, 1))^2 \, d\beta \geqq c_{11} \cdot d(B).$$

Section 5 was devoted to the proof of Lemma 4.1.

In the proof we used a particular case of the following very deep theorem of W. M. Schmidt in Diophantine Approximation: Suppose $y_1, y_2, \ldots, y_h$ are real algebraic numbers such that $1, y_1, \ldots, y_h$ are linearly independent over the rationals, and suppose $c > 1$. There are only finitely many positive integers q with

$$q^c \cdot \|y_1 \cdot q\| \cdot \|y_2 \cdot q\| \ldots \|y_h \cdot q\| < 1. \tag{0.2}$$

Unfortunately, one can at present not give an upper bound $B = B(y_1, y_2, \ldots, y_h, c)$ for solutions q of (0.2). Hence, Schmidt's theorem is "ineffective". This is the reason that our threshold constant c_0 is also "ineffective".

The object of this paper is to prove Lemma 4.2. In the following two sections (Sections 6—7, for the sake of unity), we shall prove the lower and the upper bounds, respectively.

6. Proof of Lemma 4.2 — Lower bound

Let $\varrho_B(\tau)$, $0 \leqq \tau < 2\pi$ denote the radius of curvature of B (here τ denotes the direction of the normal vector). It is well known that $\int_0^{2\pi} \varrho_B(\tau) d\tau = \text{perimeter}(B)$.

If B is an ellipse, then $f_B(\tau, 1) \approx 2(2\varrho_B(\tau))^{1/2}$, hence $\int_0^{2\pi} (f_B(\tau, 1))^2 d\tau \approx 8 \int_0^{2\pi} \varrho_B(\tau) d\tau = 8 \cdot \text{perimeter}(B)$, and Lemma 4.2 follows.

Unfortunately, in the general case (i.e., when B is an arbitrary convex region), the functions $f_B(\tau, 1)$ and $(\varrho_B(\tau))^{1/2}$ $(0 \leqq \tau < 2\pi)$ can have quite different order of magnitudes, and this natural approach breaks down. We were unable to find a simple proof; the following one is rather lengthy. We hope that the reader can essentially simplify it.

We start with some terminology. Let B be a convex compact region on the X_1X_2-plane$=\mathbf{R}^2$. Let $\Gamma(B)$ denote the boundary of B. Let $\mathbf{x}', \mathbf{x}'' \in \mathbf{R}^2$ be two distinct points. Denote by $[\mathbf{x}', \mathbf{x}'']$ the *line segment* joining $\mathbf{x}'$ and $\mathbf{x}''$, i.e., $[\mathbf{x}', \mathbf{x}''] = \{\alpha \cdot \mathbf{x}' + (1-\alpha)\mathbf{x}'' : 0 \leqq \alpha \leqq 1\}$. Denote by $L(\mathbf{x}', \mathbf{x}'')$ the straight line determined by $\mathbf{x}'$ and $\mathbf{x}''$ (we clearly have $[\mathbf{x}', \mathbf{x}''] \subset L(\mathbf{x}', \mathbf{x}'')$). Let $R(\mathbf{x}', \mathbf{x}'')$ denote the ray starting from $\mathbf{x}'$ and passing through $\mathbf{x}''$. The directed line $\overrightarrow{\mathbf{x}'\mathbf{x}''}$ splits the plane into a positive half-plane $\mathrm{HP}^+(\mathbf{x}', \mathbf{x}'')$ and a negative half-plane $\mathrm{HP}^-(\mathbf{x}', \mathbf{x}'')$ according as the points $\mathbf{x}', \mathbf{x}''$, any $\mathbf{y} \in \mathrm{HP}^+(\mathbf{x}', \mathbf{x}'')$ and $\mathbf{x}', \mathbf{x}''$, any $\mathbf{z} \in \mathrm{HP}^-(\mathbf{x}', \mathbf{x}'')$ are in clockwise and counter-clockwise orientation, respectively. Similarly, $\overrightarrow{\mathbf{x}', \mathbf{x}''}$ splits the perpendicular bisector of $[\mathbf{x}', \mathbf{x}'']$ into a positive part $\subset \mathrm{HP}^+(\mathbf{x}', \mathbf{x}'')$ and a negative part $\subset \mathrm{HP}^-(\mathbf{x}', \mathbf{x}'')$. Let $\tau(\mathbf{x}', \mathbf{x}'')$ be the angle between X_1^+ (i.e., the positive part of X_1-axis) and the positive part of the perpendicular bisector of $[\mathbf{x}', \mathbf{x}'']$. Let $|\mathbf{x}-\mathbf{y}| = ((x_1-y_1)^2 + (x_2-y_2)^2)^{1/2}$, $\mathbf{x} = (x_1, x_2) \in \mathbf{R}^2$, $\mathbf{y} = (y_1, y_2) \in \mathbf{R}^2$ denote the usual Euclidean distance.

Given $\mathbf{v}, \mathbf{v}', \mathbf{v}'' \in \Gamma(B)$, let $\operatorname{dist}(\mathbf{v}|\mathbf{v}', \mathbf{v}'')$ denote the Euclidean distance of the point $\mathbf{v}$ from the straight line $L(\mathbf{v}', \mathbf{v}'')$. Let

$$\delta_B(\mathbf{v}', \mathbf{v}'') = \max_{\mathbf{w} \in \Gamma_B(\mathbf{v}', \mathbf{v}'')} \operatorname{dist}(\mathbf{w}|\mathbf{v}', \mathbf{v}'')$$

where $\Gamma_B(\mathbf{v}', \mathbf{v}'') \subset \Gamma(B)$ denotes the arc starting from $\mathbf{v}'$ in counter-clockwise direction and terminating at $\mathbf{v}''$. Let $\mathbf{w} = \mathbf{w}_B(\mathbf{v}', \mathbf{v}'') \in \Gamma_B(\mathbf{v}', \mathbf{v}'')$ be (one of the points) defined by the equation $\operatorname{dist}(\mathbf{w}|\mathbf{v}', \mathbf{v}'') = \delta_B(\mathbf{v}', \mathbf{v}'')$.

Let $r(B)$, $d(B)$ and $l(B)$ denote the radius of the largest inscribed circle of B, the diameter of B and the perimeter of B, respectively.

For any $\beta \in [0, 2\pi)$, write $\mathbf{e}(\beta) = (\cos\beta, \sin\beta)$. Let $h_B(\beta, x)$ be the length of the chord $\{\mathbf{x} \in B \colon \mathbf{x} \cdot \mathbf{e}(\beta) = x\}$. Let $M_\beta^+ = \sup\{x \in \mathbf{R} \colon h_B(\beta, x) > 0\}$ and $M_\beta^- = \inf\{x \in \mathbf{R} \colon h_B(\beta, x) > 0\}$. Note that $M_\beta^+ = -M_{\beta+\pi}^-$. Note further that the straight lines passing through the points $M_\beta^+ \cdot \mathbf{e}(\beta)$ and $M_\beta^- \cdot \mathbf{e}(\beta)$, resp. and having angle β with the X_2-axis are tangent to B.

For later purposes we mention here two simple consequences of the convexity of B: For arbitrary $\beta \in [0, 2\pi)$,

$$\text{(6.1)} \qquad \text{if} \quad 0 < y \leqq \delta \quad \text{then} \quad h_B(\beta, M_\beta^+ - y) \geqq \frac{y}{\delta} \cdot h_B(\beta, M_\beta^+ - \delta),$$

$$\text{(6.2)} \qquad \text{if} \quad 0 < \delta \leqq y \leqq r(B) \quad \text{then}$$

$$h_B(\beta, M_\beta^+ - y) \geqq \frac{(M_\beta^+ - M_\beta^-) - y}{(M_\beta^+ - M_\beta^-) - \delta} \cdot h_B(\beta, M_\beta^+ - \delta) \geqq$$

$$\geqq \frac{2r(B) - y}{2r(B) - \delta} \cdot h_B(\beta, M_\beta^+ - \delta) \geqq \frac{1}{2} h_B(\beta, M^+ - \delta).$$

We shall also use the following well known fact:

$$\text{(6.3)} \qquad \text{if } A_1 \subset A_2 \text{ are compact convex regions then } l(A_1) \leqq l(A_2).$$

For any $\tau \in [0, 2\pi)$ and $\delta \in [0, r(B)]$, let $\mathbf{v}'(\tau, \delta)$ and $\mathbf{v}''(\tau, \delta)$ be two points on $\Gamma(B)$ such that $\tau(\mathbf{v}'(\tau, \delta), \mathbf{v}''(\tau, \delta)) = \tau$ and $\delta_B(\mathbf{v}'(\tau, \delta), \mathbf{v}''(\tau, \delta)) = \delta$. Write $f_B(\tau, \delta) = |\mathbf{v}'(\tau, \delta) - \mathbf{v}''(\tau, \delta)|$. We clearly have

$$\text{(6.4)} \qquad f_B(\tau, \delta) = h_B(\tau, M_\tau^+ - \delta).$$

Let $c_9 = 1000$. Our aim is to show that if $r(B) \geqq c_9$ then

$$\text{(6.5)} \qquad \int_0^{2\pi} (f_B(\tau, 1))^2 \, d\tau \geqq c_{11} \cdot d(B).$$

We start the proof of (6.5) with the following construction. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be points on the arc $\Gamma(B)$ in counter-clockwise direction such that $\tau(\mathbf{v}_1, \mathbf{v}_2) = 0$, $\delta_B(\mathbf{v}_i, \mathbf{v}_{i+1}) = 1$ for $1 \leqq i \leqq n-1$ and $\delta_B(\mathbf{v}_n, \mathbf{v}_1) \leqq 1$. Let $l_i = |\mathbf{v}_i - \mathbf{v}_{i+1}|$, $1 \leqq i \leqq n-1$ and $l_n = |\mathbf{v}_n - \mathbf{v}_1|$; $\tau_i = \tau(\mathbf{v}_i, \mathbf{v}_{i+1})$, $1 \leqq i \leqq n-1$ and $\tau_n = \tau(\mathbf{v}_n, \mathbf{v}_1)$. Clearly $0 = \tau_1 < \tau_2 < \ldots < \tau_n < 2\pi$.

An outline of the proof of (6.5) is as follows. Using a simple greedy algorithm we shall choose indices $1\leqq j_1<j_2<j_3<\ldots<n$ such that the angle-intervals $\left[\tau_{j_i}-\frac{1}{l_{j_i}}, \tau_{j_i}+\frac{1}{l_{j_i}}\right]$ are pairwise disjoint; $f_B(\tau, 1)>\text{const}\cdot l_{j_i}$ for all $\tau\in\left[\tau_{j_i}-\frac{1}{l_{j_i}}, \tau_{j_i}+\frac{1}{l_{j_i}}\right]$ and $l_{j_1}+l_{j_2}+l_{j_3}+\ldots>\text{const}\cdot\left(\sum_{i=1}^{n} l_i\right)>\text{const}\cdot d(B)$. We then have

$$\int_0^{2\pi} (f_B(\tau, 1))^2\,d\tau \geqq \sum_{j_i} \int_{\tau_{j_i}-\frac{1}{l_{j_i}}}^{\tau_{j_i}+\frac{1}{l_{j_i}}} (f_B(\tau, 1))^2\,d\tau > \text{const}\cdot d(B),$$

and (6.5) follows.

For notational convenience, write

$$\mathbf{v}_{k\cdot n+i} = \mathbf{v}_i, \quad k\in\mathbf{Z}, \quad 1\leqq i\leqq n$$

$$l_i = |\mathbf{v}_i-\mathbf{v}_{i+1}|, \quad i\in\mathbf{Z} \quad \text{and}$$

$$\tau_{k\cdot n+i} = \tau_i+2\pi\cdot k, \quad k\in\mathbf{Z}, \quad 1\leqq i\leqq n.$$

Let $\mathbf{w}_i=\mathbf{w}_B(\mathbf{v}_i, \mathbf{v}_{i+1})$, i.e., $\mathbf{w}_i$ is defined by the equation $\text{dist}(\mathbf{w}_i|\mathbf{v}_i, \mathbf{v}_{i+1})=\delta_B(\mathbf{v}_i, \mathbf{v}_{i+1})$. Let $\tau_i^*=\tau(\mathbf{v}_i, \mathbf{w}_i)$ and $\tau_i^{**}=\tau(\mathbf{w}_i, \mathbf{v}_{i+1})$. Let $\varphi_i=\tau_i^{**}-\tau_i^*=\pi-\text{angle}(\mathbf{v}_i, \mathbf{w}_i, \mathbf{v}_{i+1})$. Since $\tau_{i-1}<\tau_i^*<\tau_i<\tau_i^{**}<\tau_{i+1}$, we have

$$\tau_{i+1}-\tau_{i-1} \geqq \varphi_i. \tag{6.6}$$

For $1\leqq i\leqq n-1$, let $\psi_i\in(0, \pi)$ be the solution of the equation

$$\tan\left(\frac{\psi_i}{2}\right) = \frac{1}{\frac{1}{2}|\mathbf{v}_i-\mathbf{v}_{i+1}|} = \frac{2}{l_i}. \tag{6.7}$$

Simple geometric consideration shows that for $1\leqq i\leqq n-1$,

$$\varphi_i \geqq \psi_i. \tag{6.8}$$

Since $x\geqq\min\left\{\frac{\pi}{4}\cdot\tan(x), \frac{\pi}{4}\right\}$ for $0\leqq x\leqq\frac{\pi}{2}$, from (6.6)—(6.8) it follows that for $1\leqq i\leqq n-1$,

$$\tau_{i+1}-\tau_{i-1} \geqq \varphi_i \geqq \psi_i = 2\cdot\frac{\psi_i}{2} \geqq 2\min\left\{\frac{\pi}{4}\tan\left(\frac{\psi_i}{2}\right), \frac{\pi}{4}\right\} = \tag{6.9}$$

$$= \min\left\{\frac{\pi}{l_i}, \frac{\pi}{2}\right\} = \frac{\pi}{\max\{l_i, 2\}}.$$

We require

Lemma 6.1. *Suppose that there exists* $i\in[1, n]$ *such that* $l_i=|\mathbf{v}_i-\mathbf{v}_{i+1}|\geqq\frac{2}{c_9}d(B)=\frac{d(B)}{500}$. *Then inequality* (6.5) *holds with* $c_{11}=\frac{1}{2(c_9)^3}=\frac{1}{2\cdot 10^9}$.

Proof. Let $\tau\in\left[\tau_i-\frac{1}{l_i}, \tau_i\right]$. Let $\mathbf{u}\in HP^-(\mathbf{v}_i, \mathbf{v}_{i+1})$ be that point for which $\mathbf{v}_i, \mathbf{u}, \mathbf{v}_{i+1}$ forms a right-angle triangle, angle $(\mathbf{v}_i, \mathbf{u}, \mathbf{v}_{i+1})=\frac{\pi}{2}$ and $\tau(\mathbf{u}, \mathbf{v}_{i+1})=\tau$. Let $\mathbf{v}\in\Gamma(B)$ be defined by $R(\mathbf{v}_{i+1}, \mathbf{u})\cap\Gamma(B)=\{\mathbf{v}_{i+1}, \mathbf{v}\}$. Let $\mathbf{w}=\mathbf{w}_B(\mathbf{v}, \mathbf{v}_{i+1})$, i.e., $\operatorname{dist}(\mathbf{w}|\mathbf{v}, \mathbf{v}_{i+1})=\delta_B(\mathbf{v}, \mathbf{v}_{i+1})$. For convenience, write $d(\tau)=\operatorname{dist}(\mathbf{w}|\mathbf{v}, \mathbf{v}_{i+1})$. Observe that

$$f_B(\tau, d(\tau)) = |\mathbf{v}-\mathbf{v}_{i+1}|. \tag{6.10}$$

We are going to give an upper bound to $d(\tau)$. Let

$$\theta = \begin{cases}\tau(\mathbf{v}_i, \mathbf{w}), & \text{if } \mathbf{w}\in\Gamma_B(\mathbf{v}_i, \mathbf{v}_{i+1})\\ \tau(\mathbf{w}, \mathbf{v}_i), & \text{if } \mathbf{w}\notin\Gamma_B(\mathbf{v}_i, \mathbf{v}_{i+1}).\end{cases}$$

We have

$$\begin{aligned} d(\tau) &= |\mathbf{u}-\mathbf{v}_i|+|\mathbf{v}_i-\mathbf{w}|\cdot\sin(\theta-\tau) = \\ &= |\mathbf{v}_i-\mathbf{v}_{i+1}|\cdot\sin(\tau_i-\tau)+|\mathbf{v}_i-\mathbf{w}|\cdot\sin(\theta-\tau). \end{aligned} \tag{6.11}$$

On the other hand, by the "maximum distance" property of $\mathbf{w}_i=\mathbf{w}_B(\mathbf{v}_i, \mathbf{v}_{i+1})$,

$$|\mathbf{v}_i-\mathbf{w}|\cdot\sin(\theta-\tau_i) \leqq \delta_B(\mathbf{v}_i, \mathbf{v}_{i+1}) \leqq 1. \tag{6.12}$$

Since $|\sin\alpha-\sin\beta|\leqq|\alpha-\beta|$ and $|\sin\alpha|\leqq\alpha$, by (6.11) and (6.12) we have

$$\begin{aligned} d(\tau) &= |\mathbf{v}_i-\mathbf{w}|\cdot\big(\sin(\theta-\tau)-\sin(\theta-\tau_i)\big)+|\mathbf{v}_i-\mathbf{w}|\cdot\sin(\theta-\tau_i)+ \\ &+|\mathbf{v}_i-\mathbf{v}_{i+1}|\cdot\sin(\tau_i-\tau) \leqq |\mathbf{v}_i-\mathbf{w}|\cdot|\tau_i-\tau|+1+|\mathbf{v}_i-\mathbf{v}_{i+1}|\cdot|\tau_i-\tau|. \end{aligned} \tag{6.13}$$

Since $|\mathbf{v}_i-\mathbf{v}_{i+1}|=l_i$, $|\mathbf{v}_i-\mathbf{w}|\leqq d(B)$, $0\leqq\tau_i-\tau\leqq\frac{1}{l_i}$ and by hypothesis $l_i\geqq\frac{1}{c_9}d(B)$, by (6.13) we have

$$d(\tau) \leqq d(B)\cdot\frac{1}{l_i}+1+1 \leqq \frac{c_9}{2}+2 < c_9. \tag{6.14}$$

Let $C(r, \mathbf{c})=\{\mathbf{x}\in\mathbf{R}^2\colon |\mathbf{x}-\mathbf{c}|=r\}$ be the largest inscribed circle of B, i.e., $r=r(B)$ is the radius and $\mathbf{c}=\mathbf{c}(B)$ is the centre. Let $C^*=C(r-1, \mathbf{c})=\{\mathbf{y}\in\mathbf{R}^2\colon |\mathbf{y}-\mathbf{c}|=r-1\}$. Let T_1 and T_2 be the two tangents from $\mathbf{v}_i$ to C^*. Let γ denote the angle between T_1 and T_2. We have $\gamma\geqq\omega$ where ω is defined by the equation

$$\sin\left(\frac{\omega}{2}\right) = \frac{r(B)-1}{d(B)}.$$

Thus we get (we also use the fact that $r(B)\geqq c_9$)

$$\gamma \geqq 2\cdot\frac{\omega}{2} \geqq 2\cdot\sin\left(\frac{\omega}{2}\right) = \frac{2r(B)-2}{d(B)} > \frac{r(B)}{d(B)}. \tag{6.15}$$

Since $\delta_B(\mathbf{v}_i, \mathbf{v}_{i+1})\leqq 1$, we obtain that the ray $R(\mathbf{v}_{i+1}, \mathbf{u})$ certainly intersects both tangents T_1 and T_2. Let $\mathbf{t}_1=R(\mathbf{v}_{i+1}, \mathbf{u})\cap T_1$ and $\mathbf{t}_2=R(\mathbf{v}_{i+1}, \mathbf{u})\cap T_2$. We can assume that $\mathbf{t}_1\in[\mathbf{v}_{i+1}, \mathbf{t}_2]$. Then we have

$$[\mathbf{v}_{i+1}, \mathbf{v}] \cap B \supseteqq [\mathbf{v}_{i+1}, \mathbf{t}_2]. \tag{6.16}$$

Moreover, by (6.15) we have

$$\text{angle}(\mathbf{t}_2, \mathbf{v}_i, \mathbf{v}_{i+1}) \geqq \gamma \geqq \frac{r(B)}{d(B)} \geqq \frac{c_9}{d(B)}. \tag{6.17}$$

On the other hand,

$$\text{angle}(\mathbf{t}_2, \mathbf{v}_{i+1}, \mathbf{v}_i) = \tau_i - \tau \leqq \frac{1}{l_i} \leqq \frac{c_9}{2d(B)}. \tag{6.18}$$

Combining (6.17) and (6.18), we obtain that angle $(\mathbf{t}_2, \mathbf{v}_i, \mathbf{v}_{i+1}) >$ angle $(\mathbf{t}_2, \mathbf{v}_{i+1}, \mathbf{v}_i)$. Hence $|\mathbf{v}_{i+1}-\mathbf{t}_2| > |\mathbf{t}_2-\mathbf{v}_i|$, and so we have $|\mathbf{v}_{i+1}-\mathbf{t}_2| \geqq |\mathbf{v}_i-\mathbf{v}_{i+1}|-|\mathbf{t}_2-\mathbf{v}_i| > |\mathbf{v}_i-\mathbf{v}_{i+1}|-|\mathbf{v}_{i+1}-\mathbf{t}_2|$, that is,

$$|\mathbf{v}_{i+1}-\mathbf{t}_2| > \frac{1}{2}|\mathbf{v}_i-\mathbf{v}_{i+1}| = \frac{l_i}{2}. \tag{6.19}$$

Now by (6.10), (6.16) and (6.19) we get

$$f_B(\tau, d(\tau)) \geqq |\mathbf{v}_{i+1}-\mathbf{t}_2| > \frac{l_i}{2} \quad \text{provided} \quad 0 \leqq \tau_i - \tau \leqq \frac{1}{l_i}. \tag{6.20}$$

Since $r(B) \geqq c_9$, from (6.1), (6.2), (6.4), (6.14) and (6.20) it follows that $f_B(\tau, 1) \geqq \geqq \frac{1}{c_9} \cdot f_B(\tau, d(\tau)) > \frac{l_i}{2c_9}$ provided $0 \leqq \tau_i - \tau \leqq \frac{1}{l_i}$. Thus we conclude that

$$\int_0^{2\pi} (f_B(\tau, 1))^2 \, d\tau \geqq \int_{\tau_i - \frac{1}{l_i}}^{\tau_i} (f_B(\tau, 1))^2 \, d\tau > \frac{1}{l_i} \cdot \left(\frac{l_i}{2c_9}\right)^2 = \frac{l_i}{4(c_9)^2} \geqq \frac{d(B)}{2(c_9)^3}.$$

Lemma 6.1 follows. ▮

From now one we can assume that $l_i < \frac{2}{c_9} d(B) = \frac{d(B)}{500}$ for all $i \in [1, n]$. Let P denote the convex polygon of vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Clearly $P \subset B$. Next we need

Lemma 6.2. *We have* $l(P) > \frac{1}{4} l(B)$.

Proof. Let B_i denote the convex region bordered by the chord $[\mathbf{v}_i, \mathbf{v}_{i+1}]$ and the arc $\Gamma_B(\mathbf{v}_i, \mathbf{v}_{i+1})$. Let $I^* = \left\{i \in [1, n]\colon \text{angle}(\mathbf{v}_i, \mathbf{w}_i, \mathbf{v}_{i+1}) \geqq \frac{\pi}{2}\right\}$ and $I^{**} = \{i \in [1, n]\colon$ angle $(\mathbf{v}_i, \mathbf{w}_i, \mathbf{v}_{i+1}) < \frac{\pi}{2}\Big\}$.

First assume that $i \in I^*$. Since $\delta_B(\mathbf{v}_i, \mathbf{v}_{i+1}) \leqq 1$, the region B_i can be covered by a rectangle of size $l_i \times 1$, where $l_i = |\mathbf{v}_i - \mathbf{v}_{i+1}|$. Thus by (6.3) we have

$$\sum_{i \in I^*} (l_i + 2) \geqq \sum_{i \in I^*} \text{length}(\Gamma_B(\mathbf{v}_i, \mathbf{v}_{i+1})). \tag{6.21}$$

Next assume that $i \in I^{**}$. Let L_1 denote the straight line parallel to $[\mathbf{v}_i, \mathbf{v}_{i+1}]$ and passing through the centre $\mathbf{c}$ of the largest inscribed circle $C(r, \mathbf{c})$ of B. Let L_2 denote the straight line parallel to $[\mathbf{v}_i, \mathbf{v}_{i+1}]$ and passing through $\mathbf{w}_i$. Let $\{\mathbf{c}', \mathbf{c}''\} = C(r, \mathbf{c}) \cap L_1$. We may assume, without loss of generality, that $[\mathbf{c}', \mathbf{v}_i] \cap [\mathbf{c}'', \mathbf{v}_{i+1}] = \emptyset$.

Further, let $\mathbf{w}'=L_2\cap L(\mathbf{c}', \mathbf{v}_i)$ and $\mathbf{w}''=L_2\cap L(\mathbf{c}'', \mathbf{v}_{i+1})$. Note that the region B_i is contained in the trapezium $\mathbf{v}_i, \mathbf{v}_{i+1}, \mathbf{w}'', \mathbf{w}'$, and so we have length $(\Gamma_B(\mathbf{v}_i, \mathbf{v}_{i+1}))\leqq \leqq |\mathbf{v}_{i+1}-\mathbf{w}''|+|\mathbf{w}''-\mathbf{w}'|+|\mathbf{w}'-\mathbf{v}_i|$. Let $\mathbf{v}=[\mathbf{v}_i, \mathbf{v}_{i+1}]\cap[\mathbf{w}_i, \mathbf{c}]$. Observe that

$$\frac{|\mathbf{w}_i-\mathbf{v}|}{|\mathbf{v}-\mathbf{c}|} \leqq \frac{\delta_B(\mathbf{v}_i, \mathbf{v}_{i+1})}{r(B)} \leqq \frac{1}{c_9} = \frac{1}{1000}.$$

Since $|\mathbf{v}-\mathbf{c}|\leqq d(B)<\frac{l(B)}{2}$, we obtain $|\mathbf{w}_i-\mathbf{v}|<\frac{l(B)}{2000}$. We clearly have

$$|\mathbf{w}'-\mathbf{w}''| \leqq |\mathbf{v}_i-\mathbf{v}_{i+1}| = l_i,$$

$$|\mathbf{v}_{i+1}-\mathbf{w}''| \leqq |\mathbf{w}''-\mathbf{w}_i|+|\mathbf{w}_i-\mathbf{v}|+|\mathbf{v}-\mathbf{v}_{i+1}|,$$

$$|\mathbf{w}'-\mathbf{v}_i| \leqq |\mathbf{v}_i-\mathbf{v}|+|\mathbf{v}-\mathbf{w}_i|+|\mathbf{w}_i-\mathbf{w}'|.$$

Summarizing, we have that length $(\Gamma_B(\mathbf{v}_i, \mathbf{v}_{i+1}))\leqq 2\frac{l(B)}{2000}+3l_i$. Since $\varphi_i=\pi-$ $-\text{angle}\,(\mathbf{v}_i, \mathbf{w}_i, \mathbf{v}_{i+1})$ and $\sum_{i=1}^{n}\varphi_i=2\pi$, it follows that card $I^{**}\leqq 3$. Hence $\sum_{i\in I^{**}}$ length $(\Gamma_B(\mathbf{v}_i, \mathbf{v}_{i+1}))\leqq 3\cdot\frac{l(B)}{1000}+3\sum_{i\in I^{**}} l_i\leqq 3\frac{l(B)}{1000}+3\cdot 3\cdot\frac{d(B)}{500}<3\frac{l(B)}{1000}+3\cdot 3\times$ $\times\frac{l(B)}{1000}=\frac{12}{1000}l(B)$. Combining this inequality with (6.21), we obtain

$$\frac{988}{1000}l(B) \leqq \sum_{i\in I^*}(l_i+2) \leqq \sum_{i=1}^{n}(l_i+2). \tag{6.22}$$

On the other hand, by (6.9) we have

$$2\pi = \sum_{i=1}^{n}\varphi_i \geqq \sum_{i=1}^{n-1}\varphi_i \geqq \sum_{i=1}^{n-1}\psi_i \geqq \sum_{i=1}^{n-1}\frac{\pi}{\max\{l_i, 2\}}, \quad \text{that is,}$$

$$\sum_{i=1}^{n-1}\frac{1}{\max\{l_i, 2\}} \leqq 2.$$

Hence card $\{i\in[1, n-1]\colon l_i\leqq 2\}\leqq 4$, and so we have card $\{i\in[1, n]\colon l_i\leqq 2\}\leqq 5$. Therefore,

$$\sum_{i=1}^{n}(l_i+2) = \sum_{i:\,l_i\leqq 2}(l_i+2)+\sum_{i:\,l_i>2}(l_i+2) \leqq \tag{6.23}$$

$$\leqq 5\cdot 4+2\sum_{i:\,l_i>2} l_i \leqq 20+2\sum_{i=1}^{n} l_i = 20+2l(P).$$

By (6.22) and (6.23) we have $l(P)\geqq\frac{494}{1000}l(B)-10$. Since $l(B)\geqq 2\pi\cdot r(B)\geqq 2\pi\cdot 1000$, we conclude that $l(P)>\frac{1}{4}l(B)$, and Lemma 6.2 follows. ∎

For any $i \in \mathbf{Z}$, let $k(i)$ be the smallest integer $k \geqq i+2$ such that

$$\sum_{j=i+1}^{k-1} l_j > \frac{d(B)}{300};$$

and let $q(i)$ be the largest integer $q \leqq i-2$ such that

$$\sum_{j=q+1}^{i-1} l_j > \frac{d(B)}{300}.$$

We say that an index $i \in [1, n-1]$ and the corresponding chord $[\mathbf{v}_i, \mathbf{v}_{i+1}]$ are "good" if $\tau_k - \tau_{i-1} < \frac{\pi}{2}$ and $\tau_{i+1} - \tau_q < \frac{\pi}{2}$, where $k = k(i)$ and $q = q(i)$, respectively. Note that if the index $i \in [1, n-1]$ is "good" then $\frac{\pi}{2} > \tau_{i+1} - \tau_{i-1}$, and so by (6.9), $l_i > 2$.

Lemma 6.3. *Let* $i \in [1, n-1]$ *be a "good" index. Then*

$$\int_{\tau_i - \frac{1}{l_i}}^{\tau_i + \frac{1}{l_i}} (f_B(\tau, 1))^2 \, d\tau > \frac{1}{64} l_i.$$

Proof. We shall use the same notation as in the proof of Lemma 6.1. Since $l_i > 2$, by (6.9) we have $\tau_i^{**} - \tau_i^* = \varphi_i \geqq \frac{\pi}{\max\{l_i, 2\}} = \frac{\pi}{l_i}$. We recall: $\tau_i^* < \tau_i < \tau_i^{**}$. Let (say) $\tau_i - \tau_i^* \geqq \frac{\pi}{2l_i}$. Let $\tau \in \left[\tau_i - \frac{1}{l_i}, \tau_i\right]$. Since $\frac{1}{l_i} < \frac{\pi}{2l_i}$, we clearly have $\tau \in (\tau_i^*, \tau_i]$. Let $\mathbf{u} \in \mathrm{HP}^-(\mathbf{v}_i, \mathbf{v}_{i+1})$ be that point for which $\mathbf{v}_i, \mathbf{u}, \mathbf{v}_{i+1}$ forms a right-angle triangle, angle $(\mathbf{v}_i, \mathbf{u}, \mathbf{v}_{i+1}) = \frac{\pi}{2}$ and $\tau(\mathbf{u}, \mathbf{v}_{i+i}) = \tau$. Let $\mathbf{v} \in \Gamma(B)$ de defined by $R(\mathbf{v}_{i+1}, \mathbf{u}) \cap \cap \Gamma(B) = \{\mathbf{v}_{i+1}, \mathbf{v}\}$. Let $\mathbf{w} = \mathbf{w}_B(\mathbf{v}, \mathbf{v}_{i+1})$, i.e., $\mathrm{dist}(\mathbf{w} | \mathbf{v}, \mathbf{v}_{i+1}) = \delta_B(\mathbf{v}, \mathbf{v}_{i+1})$. We recall: $\mathbf{w}_i = \mathbf{w}_B(\mathbf{v}_i, \mathbf{v}_{i+1})$. For convenience, write $d(\tau) = \mathrm{dist}(\mathbf{w} | \mathbf{v}, \mathbf{v}_{i+1})$. Since $\tau \leqq \tau_i$, we get $\mathbf{w} \in \Gamma_B(\mathbf{v}, \mathbf{w}_i)$. Moreover, since $\tau_i^* = \tau(\mathbf{v}_i, \mathbf{w}_i) < \tau = \tau(\mathbf{v}, \mathbf{v}_{i+1})$, it follows that $\mathbf{w} \in \Gamma_B(\mathbf{v}_i, \mathbf{v}_{i+1})$. Summarizing, we obtain $\mathbf{w} \in \Gamma_B(\mathbf{v}, \mathbf{w}_i) \cap \Gamma_B(\mathbf{v}_i, \mathbf{v}_{i+1}) = = \Gamma_B(\mathbf{v}_i, \mathbf{w}_i)$. Repeating the argument in the proof of Lemma 6.1 without any modification, we obtain inequality (6.13):

$$(6.24) \qquad d(\tau) \leqq |\mathbf{v}_i - \mathbf{w}| \cdot |\tau_i - \tau| + 1 + |\mathbf{v}_i - \mathbf{v}_{i+1}| \cdot |\tau_i - \tau|.$$

Since i is a "good" index, we have angle $(\mathbf{v}_i, \mathbf{w}, \mathbf{v}_{i+1}) \geqq \pi - (\tau_{i+1} - \tau_{i-1}) \geqq \pi - (\tau_k - \tau_i) > > \pi - \frac{\pi}{2} = \frac{\pi}{2}$. Hence

$$(6.25) \quad |\mathbf{v}_i - \mathbf{w}| \leqq |\mathbf{v}_i - \mathbf{v}_{i+1}| + \mathrm{dist}(\mathbf{w} | \mathbf{v}_i, \mathbf{v}_{i+1}) \leqq |\mathbf{v}_i - \mathbf{v}_{i+1}| + \mathrm{dist}(\mathbf{w}_i | \mathbf{v}_i, \mathbf{v}_{i+1}) =$$

$$= l_i + \delta_B(\mathbf{v}_i, \mathbf{v}_{i+1}) \leqq l_i + 1.$$

Since $\tau\in\left[\tau_i-\frac{1}{l_i},\ \tau_i\right]$ and $l_i>2$, by (6.24) and (6.25) we have

$$d(\tau)\leqq(l_i+1)\cdot\frac{1}{l_i}+1+l_i\cdot\frac{1}{l_i}=3+\frac{1}{l_i}<4. \tag{6.26}$$

We recall (6.10):

$$f_B(\tau,\ d(\tau))=|\mathbf{v}-\mathbf{v}_{i+1}|. \tag{6.27}$$

We claim that

$$[\mathbf{v}_{i+1},\ u]\subset[\mathbf{v}_{i+1},\ \mathbf{v}]. \tag{6.28}$$

Assume, in contrary, that the arc $\Gamma_B(\mathbf{v}_{i+1},\ \mathbf{v}_i)=\Gamma(B)\setminus\Gamma_B(\mathbf{v}_i,\ \mathbf{v}_{i+1})$ *does* intersect the line segment $[\mathbf{v}_{i+1},\ \mathbf{u}]$ in the point $\mathbf{v}(\neq\mathbf{v}_{i+1})$. Then there must exist a chord $[\mathbf{v}_j,\ \mathbf{v}_{j+1}]$, $j\leqq i-1$ such that $\mathbf{v}_{j+1}\in\Gamma_B(\mathbf{v},\ \mathbf{v}_i)$, $[\mathbf{v}_j,\ \mathbf{v}_{j+1}]\cap[\mathbf{u},\ \mathbf{v}_i]\neq\emptyset$ and

$$\tau_i-\tau_j\geqq\pi-\text{angle}\,(\mathbf{u},\ \mathbf{v}_i,\ \mathbf{v}_{i+1})=\frac{\pi}{2}+(\tau_i-\tau). \tag{6.29}$$

Let $\mathbf{z}_i=[\mathbf{v}_j,\ \mathbf{v}_{j+1}]\cap[\mathbf{u},\ \mathbf{v}_i]$. Let $\tilde{L}$ denote the straight line parallel to $[\mathbf{u},\ \mathbf{v}_i]$ and passing through the centre $\mathbf{c}$ of the largest inscribed circle $C(r,\ \mathbf{c})$ of B. Let $\tilde{\mathbf{v}}=L(\mathbf{v}_i,\ \mathbf{v}_{i+1})\cap\tilde{L}$, $\tilde{\mathbf{z}}=L(\mathbf{v}_j,\ \mathbf{v}_{j+1})\cap\tilde{L}$, $\mathbf{x}=L(\mathbf{v}_i,\ \mathbf{v}_{i+1})\cap L(\mathbf{v}_j,\ \mathbf{v}_{j+1})$ and $\mathbf{y}=L(\mathbf{c},\ \mathbf{x})\cap[\mathbf{u},\ \mathbf{v}_i]$. Observe that $\mathbf{x}$, $\tilde{\mathbf{v}}$, $\tilde{\mathbf{z}}$ and $\mathbf{x}$, $\mathbf{v}_i$, $\mathbf{z}_i$ are *homothetic* triangles. Thus we have

$$\frac{|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}-\mathbf{c}|}=\frac{|\mathbf{z}_i-\mathbf{v}_i|}{|\tilde{\mathbf{z}}-\tilde{\mathbf{v}}|}. \tag{6.30}$$

Since

$$|\mathbf{u}-\mathbf{v}_i|=|\mathbf{v}_i-\mathbf{v}_{i+1}|\cdot\sin(\tau_i-\tau)\leqq l_i\cdot\sin\left(\frac{1}{l_i}\right)\leqq 1, \tag{6.31}$$

we have

$$|\mathbf{z}_i-\mathbf{v}_i|\leqq|\mathbf{u}-\mathbf{v}_i|\leqq 1. \tag{6.32}$$

Moreover, we have

$$|\tilde{\mathbf{z}}-\tilde{\mathbf{v}}|\geqq 2r(B)-2\geqq 2c_9-2. \tag{6.33}$$

Thus, by (6.30)—(6.33),

$$\frac{|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}-\mathbf{c}|}\leqq\frac{1}{2c_9-2},$$

and so we have

$$|\mathbf{x}-\mathbf{y}|\leqq\frac{1}{2c_9-3}\cdot|\mathbf{y}-\mathbf{c}|. \tag{6.34}$$

By (6.31),

$$|\mathbf{y}-\mathbf{c}|\leqq|\mathbf{c}-\mathbf{v}_i|+|\mathbf{v}_i-\mathbf{y}|\leqq d(B)+|\mathbf{u}-\mathbf{v}_i|\leqq d(B)+1. \tag{6.35}$$

Combining (6.34) and (6.35), we have

$$|\mathbf{x}-\mathbf{y}|\leqq\frac{1}{2c_9-3}\cdot(d(B)+1). \tag{6.36}$$

On the other hand, by (6.32), $|\mathbf{z}_i-\mathbf{y}|+|\mathbf{y}-\mathbf{v}_i|=|\mathbf{z}_i-\mathbf{v}_i|\leqq 1$, thus we have $|\mathbf{v}_i-\mathbf{x}|\leqq$

$\leqq |\mathbf{y}-\mathbf{x}|+|\mathbf{v}_i-\mathbf{y}| \leqq |\mathbf{x}-\mathbf{y}|+1$ and $|\mathbf{z}_i-\mathbf{x}| \leqq |\mathbf{y}-\mathbf{x}|+|\mathbf{z}_i-\mathbf{y}| \leqq |\mathbf{x}-\mathbf{y}|+1$.
Hence, by (6.36)

$$(6.37)\qquad |\mathbf{v}_i-\mathbf{x}|+|\mathbf{z}_i-\mathbf{x}| \leqq 2|\mathbf{x}-\mathbf{y}|+2 \leqq \frac{2}{2c_9-3}(d(B)+1)+2 \leqq$$

$$\leqq \frac{2}{2c_9-3}(d(B)+1)+\frac{d(B)}{c_9} < \frac{3}{c_9}d(B) < \frac{d(B)}{300}.$$

Thus by (6.3) and (6.37) we have

$$(6.38)\qquad \sum_{t=j+1}^{i-1} l_t \leqq \text{length}\,(\Gamma_B(\mathbf{v}_{j+1}, \mathbf{v}_i)) \leqq \text{length}\,(\Gamma_B(\mathbf{z}_i, \mathbf{x}_i)) \leqq$$

$$\leqq |\mathbf{v}_i-\mathbf{x}|+|\mathbf{z}_i-\mathbf{x}| < \frac{d(B)}{300}.$$

Since by (6.29), $\tau_i-\tau_j \geqq \frac{\pi}{2}+(\tau_i-\tau) \geqq \frac{\pi}{2}$, inequality (6.38) contradicts to the hypothesis that the index i is "good". This proves relation (6.28).

Now by (6.27) and (6.28), we have with $\tau \in \left[\tau_i-\frac{1}{l_i},\, \tau_i\right]$,

$$(6.39)\quad f_B(\tau, d(\tau)) \geqq |\mathbf{v}_{i+1}-\mathbf{u}| = l_i \cdot \cos(\tau_i-\tau) \geqq l_i \cdot \cos\left(\frac{1}{l_i}\right) \geqq l_i \cdot \cos\left(\frac{1}{2}\right) > \frac{l_i}{2}.$$

Summarizing, from (6.1), (6.2), (6.4), (6.26) and (6.39) it follows that $f_B(\tau, 1) \geqq \frac{1}{4} f_B(\tau, d(\tau)) > \frac{l_i}{8}$ whenever $\tau \in \left[\tau_i-\frac{1}{l_i},\, \tau_i\right]$. Hence

$$\int_{\tau_i-\frac{1}{l_i}}^{\tau_i+\frac{1}{l_i}} (f_B(\tau, 1))^2\, d\tau > \frac{1}{l_i} \cdot \left(\frac{l_i}{8}\right)^2 = \frac{l_i}{64},$$

and Lemma 6.3 follows. ▮

We recall:

$$\max_{1 \leqq i \leqq n} l_i < \frac{2}{c_9} \cdot d(B) = \frac{d(B)}{500}.$$

Since by Lemma 6.2, $l(P) > \frac{1}{4} l(B)$, we have

$$(6.40)\qquad \max_{1 \leqq i \leqq n} l_i < \frac{2}{c_9} d(B) < \frac{1}{c_9} l(B) < \frac{4}{c_9} l(P) < \frac{l(P)}{150}.$$

Hence one can partition the interval $[1, n-1]$ into subintervals $I_1, I_2, \dots, I_m$ such that

$$(6.41)\qquad 2\frac{l(P)}{150} > \sum_{j \in I_\nu} l_j \geqq \frac{l(P)}{150} \quad \text{for all} \quad \nu \in [1, m].$$

Let $I_\nu=\{i_{\nu-1}+1, i_{\nu-1}+2, \ldots, i_\nu\}$; $i_0=0<i_1<i_2<\ldots<i_m=n-1$. Since $(\tau_{i_1}-\tau_{i_0})+$ $+(\tau_{i_2}-\tau_{i_1})+\ldots+(\tau_{i_m}-\tau_{i_{m-1}})=\tau_{n-1}-\tau_0=\tau_{n-1}-(\tau_n-2\pi)=2\pi-(\tau_n-\tau_{n-1})<2\pi$, there are at most 11 indices i_j such that

$$\tau_{i_j}-\tau_{i_{j-1}} \geqq \frac{\pi}{6}. \tag{6.42}$$

We call them "forbidden" indices.

We say that an index i_ν is "nice" if there is *no* "forbidden" index in the set $\{i_{\nu-2}, i_{\nu-1}, i_\nu, i_{\nu+1}, i_{\nu+2}\}$ (let $i_{m+1}=i_1$). Clearly there are at least $m-55$ "nice" indices among i_ν, $1\leqq\nu\leqq m$. Since by Lemma 6.2, $l(P)>\frac{1}{4}\,l(B)>\frac{1}{2}\,d(B)$, we have

$$\frac{l(P)}{150}>\frac{d(B)}{300}. \tag{6.43}$$

From (6.41)—(6.43) it follows that if i_ν is "nice" and $j\in I_\nu$, then j is "good". Since there are at least $m-55$ "nice" indices among $\{i_\nu\colon 1\leqq\nu\leqq m\}$, by (6.40) and (6.41) we have

$$\sum_{\substack{1\leqq j\leqq n-1:\\ j \text{ is "good"}}} l_j \geqq \sum_{\substack{1\leqq \nu\leqq m:\\ \nu \text{ is "nice"}}} \sum_{j\in I_\nu} l_j \geqq \tag{6.44}$$

$$\sum_{\nu=1}^{m}\sum_{j\in I_\nu} l_j-55\cdot 2\,\frac{l(P)}{150}=l(P)-l_n-\frac{11}{15}\cdot l(P)>\left(1-\frac{1}{250}-\frac{11}{15}\right)\cdot l(P)>\frac{l(P)}{5}.$$

Next we are going to find a lot of "good" indices $i\in[1, n-1]$ such that the angle-intervals $\left[\tau_i-\frac{1}{l_i}, \tau_i+\frac{1}{l_i}\right]$ (mod 2π) are pairwise disjoint.

Lemma 6.4. *Let there be given t points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_t$ on the unit circle $C=C(1, \mathbf{0})=$ $=\{\mathbf{x}\in\mathbf{R}^2\colon |\mathbf{x}|=1\}$. Let $\alpha_1\leqq\frac{1}{2}$, $\alpha_2\leqq\frac{1}{2}, \ldots, \alpha_t\leqq\frac{1}{2}$ be positive real numbers. Let*

$[\mathbf{p}_i-\alpha_i, \mathbf{p}_i+\alpha_i]=\{\mathbf{q}\in C\colon$ the length of the arc joining $\mathbf{q}$ and $\mathbf{p}_i$ is $\leqq\alpha_i\}$.

Suppose that for any $i\in[1, t]$,

$$\operatorname{card}\{j\in[1, t]\colon \mathbf{p}_j\in[\mathbf{p}_i-\alpha_i, \mathbf{p}_i+\alpha_i]\}\leqq 2. \tag{6.45}$$

Let $i_0\in[1, t]$ be an index such that $\alpha_{i_0}=\min_{1\leqq i\leqq t}\alpha_i$. Write $J=J(i_0)=\{j\in[1, t]\colon$ $[\mathbf{p}_j-\alpha_j, \mathbf{p}_j+\alpha_j]\cap[\mathbf{p}_{i_0}-\alpha_{i_0}, \mathbf{p}_{i_0}+\alpha_{i_0}]\neq\emptyset\}$. Then

$$\frac{1}{\alpha_{i_0}}\geqq\frac{1}{14}\sum_{j\in J}\frac{1}{\alpha_j}.$$

Proof. Let $\mathbf{p}'$, $\mathbf{p}''$ denote the end-points of the arc $[\mathbf{p}_{i_0}-\alpha_{i_0}, \mathbf{p}_{i_0}+\alpha_{i_0}]$. The straight line $L(\mathbf{p}_i, \mathbf{0})$ passing through $\mathbf{p}_i$ and the centre $\mathbf{0}$ of C splits C into two half-circles C' and C''. Suppose that $\mathbf{p}'\in C'$ and $\mathbf{p}''\in C''$. Let

$$J'=\{j\in[1, t]\colon \mathbf{p}_j\in C'\setminus[\mathbf{p}_{i_0}-\alpha_{i_0}, \mathbf{p}_{i_0}+\alpha_{i_0}] \text{ and } \mathbf{p}'\in[\mathbf{p}_j-\alpha_j, \mathbf{p}_j+\alpha_j]\}$$

and

$$J''=\{j\in[1, t]\colon \mathbf{p}_j\in C''\setminus[\mathbf{p}_{i_0}-\alpha_{i_0}, \mathbf{p}_{i_0}+\alpha_{i_0}] \text{ and } \mathbf{p}''\in[\mathbf{p}_j-\alpha_j, \mathbf{p}_j+\alpha_j]\}.$$

From (6.45) easily follows that $\operatorname{card}\{j\in J'\colon \mathbf{p}_j\in[\mathbf{p}'-\alpha_{i_0}, \mathbf{p}'+\alpha_{i_0}]\}\leqq 2$. Similarly, for any integer $k\geqq 1$ we have $\operatorname{card}\{j\in J'\colon \mathbf{p}_j\in[\mathbf{p}'-2^k\cdot\alpha_{i_0}, \mathbf{p}'+2^k\cdot\alpha_{i_0}]\setminus[\mathbf{p}'-2^{k-1}\times$ $\times\alpha_{i_0}, \mathbf{p}'+2^{k-1}\cdot\alpha_{i_0}]\}\leqq 2$. Hence

(6.46) $$\sum_{j\in J'}\frac{1}{\alpha_j}\leqq 2\cdot\frac{1}{\alpha_{i_0}}+2\cdot\frac{1}{\alpha_{i_0}}+2\cdot\frac{1}{2\alpha_{i_0}}+2\cdot\frac{1}{4\alpha_{i_0}}+2\cdot\frac{1}{8\alpha_{i_0}}+\ldots=\frac{6}{\alpha_{i_0}}.$$

Similarly,

(6.47) $$\sum_{j\in J''}\frac{1}{\alpha_j}\leqq\frac{6}{\alpha_{i_0}}.$$

Finally, by (6.45)—(6.46) we have

$$\sum_{j\in J}\frac{1}{\alpha_j}\leqq\sum_{j\in J'}\frac{1}{\alpha_j}+\sum_{j\in J''}\frac{1}{\alpha_j}+2\cdot\frac{1}{\alpha_{i_0}}\leqq\frac{14}{\alpha_{i_0}},$$

and Lemma 6.4 follows. ∎

Now let $j_1\in[1, n-1]$ be a "good" index such that l_{j_1} is maximal. Throw away that "good" indices i for which the sets $\left[\tau_i-\frac{1}{l_i}, \tau_i+\frac{1}{l_i}\right]$ (mod 2π) and $\left[\tau_{j_1}-\frac{1}{l_{j_1}}, \tau_{j_1}+\frac{1}{l_{j_1}}\right]$ (mod 2π) have common point. Let j_2 be a "good" index from the remainder such that l_{j_2} is maximal. Again throw away that "good" indices i for which the sets $\left[\tau_i-\frac{1}{l_i}, \tau_i+\frac{1}{l_i}\right]$ (mod 2π) and $\left[\tau_{j_2}-\frac{1}{l_{j_2}}, \tau_{j_2}+\frac{1}{l_{j_2}}\right]$ (mod 2π) have common point. Let j_3 be a "good" index from the remainder such that l_{j_3} is maximal, and so forth. Repeating this simple greedy algorithm we obtain a sequence $j_1, j_2, j_3, \ldots$ of indices in $[1, n-1]$. We call them "special" indices. We recall: if $i\in[1, n-1]$ is "good" then $l_i>2$. Hence by (6.9) we have

(6.48) $$\tau_{i+1}-\tau_{i-1}\geqq\frac{\pi}{\max\{l_i, 2\}}=\frac{\pi}{l_i}>\frac{2}{l_i}.$$

Therefore, we can apply Lemma 6.4 in each step of the previous greedy algorithm with $\mathbf{p}_i=\mathbf{e}(\tau_i)=(\cos\tau_i, \sin\tau_i)\in C$ and $\alpha_i=\frac{1}{l_i}$. Note that relation (6.48) guarantees the fulfilment of hypothesis (6.45). By Lemma 6.4 we have

(6.49) $$\sum_{\substack{1\leqq j\leqq n-1:\\ j \text{ is "special"}}} l_j\geqq\frac{1}{14}\sum_{\substack{i\leqq j\leqq n-1:\\ j \text{ is "good"}}} l_j.$$

Combining (6.44) and (6.49), we get

(6.50) $$\sum_{\substack{1\leqq j\leqq n-1\\ j \text{ is "special"}}} l_j>\frac{l(P)}{70}.$$

Now we are ready to complete the proof of (6.5). From the construction of "special" indices above it follows that if both j and k are "special" indices in $[1, n-1]$, then the intervals $\left[\tau_j-\frac{1}{l_j}, \tau_j+\frac{1}{l_j}\right]$ and $\left[\tau_k-\frac{1}{l_k}, \tau_k+\frac{1}{l_k}\right]$ are disjoint (mod 2π). Since $\{j\in[1, n-1]: j$ is "special"$\}\subset\{j\in[1, n-1]: j$ is "good"$\}$, by Lemmas 6.2—6.3 and inequality (6.50) we have

$$\int_0^{2\pi} (f_B(\tau, 1))^2\, d\tau \geqq \sum_{\substack{1\leqq j\leqq n-1:\\ j \text{ is "special"}}} \int_{\tau_j-\frac{1}{l_j}}^{\tau_j+\frac{1}{l_j}} (f_B(\tau, 1))^2\, d\tau >$$

$$> \frac{1}{64} \sum_{\substack{1\leqq j\leqq n-1:\\ j \text{ is "special"}}} l_j > \frac{1}{64\cdot 70}\, l(P) > \frac{1}{64\cdot 70\cdot 4}\, l(B) >$$

$$> \frac{2}{64\cdot 70\cdot 4}\, d(B) > 10^{-4}\cdot d(B),$$

and (6.5) follows with $c_{11}=10^{-4}$.

7. Proof of Lemma 4.2 — Upper bound

We shall show that

$$\int_0^{2\pi} (f_B(\tau, 1))^2\, d\tau \leqq c_{10}\cdot d(B) \quad \text{provided} \quad r(B)\geqq 2. \tag{7.1}$$

We use the same notation as in Section 6. Let $D=\{\mathbf{x}\in\mathbf{R}^2: \inf_{\mathbf{y}\in B} |\mathbf{x}-\mathbf{y}|\leqq 1\}$. Clearly $D\supset B$ is a smooth compact convex region. Let $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_m$ be points on the arc $\Gamma(D)$ in counter-clockwise direction such that $\tau(\mathbf{z}_1, \mathbf{z}_2)=0$, $\delta_D(\mathbf{z}_i, \mathbf{z}_{i+1})=\delta=1-\cos(\pi/8)$, $1\leqq i\leqq m-1$ and $\delta_D(\mathbf{z}_m, \mathbf{z}_1)\leqq\delta=1-\cos(\pi/8)$. Let Q denote the convex polygon of vertices $\mathbf{z}_1, \dots, \mathbf{z}_m$. We have $B\subset Q\subset D$. Let $\theta_i=\tau(\mathbf{z}_i, \mathbf{z}_{i+1})$, $1\leqq i\leqq m-1$ and $\theta_m=\tau(\mathbf{z}_m, \mathbf{z}_1)$. Clearly $0=\theta_1<\theta_2<\dots<\theta_m<2\pi$. For notational convenience, write

$$\mathbf{z}_{k\cdot m+i}=\mathbf{z}_i, \quad k\in\mathbf{Z}, \quad 1\leqq i\leqq m \quad \text{and}$$

$$\theta_{k\cdot m+i}=\theta_i+k\cdot 2\pi, \quad k\in\mathbf{Z}, \quad 1\leqq i\leqq m.$$

Let $l_i=|\mathbf{z}_i-\mathbf{z}_{i+1}|$, $i\in\mathbf{Z}$. Let $\mathbf{w}_i=\mathbf{w}_D(\mathbf{z}_i, \mathbf{z}_{i+1})$, i.e., $\mathbf{w}_i\in\Gamma_D(\mathbf{z}_i, \mathbf{z}_{i+1})$ satisfies the equation $\operatorname{dist}(\mathbf{w}_i|\mathbf{z}_i, \mathbf{z}_{i+1})=\delta_D(\mathbf{z}_i, \mathbf{z}_{i+1})$.

We claim that

$$l_i=|\mathbf{z}_i-\mathbf{z}_{i+1}|\geqq 2\sin(\pi/8) \tag{7.2}$$

and

$$\theta_{i+1}-\theta_i\leqq\frac{\pi}{2} \quad \text{for all} \quad i\in\mathbf{Z}. \tag{7.3}$$

In order to verify (7.2) and (7.3), we introduce the following concept: A disc of radius ϱ is said to *roll in* a convex region D if for each point of $\Gamma(D)$ there is a disc of radius ϱ contained in D and containing the point. W. Blaschke [2] and D. Koutroufiotis [5] proved the following result.

Let the convex region D have curvatures. Then the following propositions are equivalent:

(7.4) The curvatures of D have upper bound $\frac{1}{\varrho}$.

(7.5) A disc of radius ϱ can roll in D.

From the above-mentioned construction of our D it follows that a disc of radius $\varrho=1$ can roll in D. The equivalence of (7.4) and (7.5) yields that the curvatures of D are $\leqq 1$. Let $T(\mathbf{z}_i)$ denote the tangent of D at $\mathbf{z}_i \in \Gamma(D)$, $i \in \mathbf{Z}$. Since $\delta_D(\mathbf{z}_i, \mathbf{z}_{i+1}) \leqq \delta = 1-\cos(\pi/8)$, we conclude that $l_i = |\mathbf{z}_i - \mathbf{z}_{i+1}| \geqq 2\sin(\pi/8)$ and

$$\text{angle}\left(T(\mathbf{z}_i), T(\mathbf{z}_{i+1})\right) \leqq \frac{\pi}{4}, \quad i \in \mathbf{Z}. \tag{7.6}$$

Now by (7.6) we have $\theta_{i+1} - \theta_i \leqq \text{angle}\left(T(\mathbf{z}_i), T(\mathbf{z}_{i+2})\right) = \text{angle}\left(T(\mathbf{z}_i), T(\mathbf{z}_{i+1})\right) + \text{angle}\left(T(\mathbf{z}_{i+1}), T(\mathbf{z}_{i+2})\right) \leqq \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$. Thus relations (7.2) and (7.3) are proved.

Next we show that if $i \not\equiv 0 \pmod m$ then

$$\theta_{i+1} - \theta_{i-1} \geqq \frac{\pi}{20 l_i}. \tag{7.7}$$

By (7.6) we have

$$\frac{\pi}{4} \geqq \text{angle}\left(T(\mathbf{z}_i), T(\mathbf{z}_{i+1})\right) \geqq \pi - \text{angle}(\mathbf{z}_i, \mathbf{w}_i, \mathbf{z}_{i+1}) \geqq \psi_i \tag{7.8}$$

where $\psi_i \in \left[0, \frac{\pi}{4}\right]$ is the solution of the equation

$$\tan(\psi_i/2) = \frac{2\delta}{l_i} = \frac{2(1-\cos(\pi/8))}{l_i} > \frac{1}{10 l_i}.$$

Since $x \geqq \frac{\pi}{4}\tan(x)$ for $0 \leqq x \leqq \pi/8$, by (7.8) we have $\theta_{i+1} - \theta_{i-1} \geqq \text{angle}\left(T(\mathbf{z}_i), T(\mathbf{z}_{i+1})\right) \geqq \psi_i = 2 \cdot \frac{\psi_i}{2} \geqq 2 \cdot \frac{\pi}{4} \tan\left(\frac{\psi_i}{2}\right) > 2 \cdot \frac{\pi}{4} \cdot \frac{1}{10\, l_i} = \frac{\pi}{20\, l_i}$, and (7.7) follows.

Let $\tau \in [0, 2\pi)$ be arbitrary but fixed. We have $\tau \in [\theta_k, \theta_{k+1})$ for some $k \in [1, m]$. Let $\mathbf{u}' = \mathbf{u}'(\tau)$ and $\mathbf{u}'' = \mathbf{u}''(\tau)$ be points on $\Gamma(Q)$ such that $\tau(\mathbf{u}', \mathbf{u}'') = \tau$ and $\delta_Q(\mathbf{u}', \mathbf{u}'') = \text{dist}(\mathbf{z}_{k+1} | \mathbf{u}', \mathbf{u}'') = 2$. Let $\mathbf{v}', \mathbf{v}'' \in \Gamma(B)$ be the end-points of the chord $[\mathbf{v}', \mathbf{v}''] = [\mathbf{u}', \mathbf{u}''] \cap B$. Write $d(\tau) = \delta_B(\mathbf{v}', \mathbf{v}'')$. Simple geometric consideration shows that $2-\delta = 2-(1-\cos(\pi/8)) \leqq d(\tau) \leqq 2$, and so we have

$$1 \leqq d(\tau) \leqq 2. \tag{7.9}$$

Clearly

$$f_B\left(\tau, d(\tau)\right) = |\mathbf{v}' - \mathbf{v}''| \leqq |\mathbf{u}' - \mathbf{u}''|. \tag{7.10}$$

There is a point $\mathbf{u}_0 \in L(\mathbf{u}', \mathbf{u}'')$ such that angle $(\mathbf{u}', \mathbf{u}_0, \mathbf{z}_{k+1}) =$ angle $(\mathbf{u}'', \mathbf{u}_0, \mathbf{z}_{k+1}) = \pi/2$. We have

$$|\mathbf{u}' - \mathbf{u}''| \leqq |\mathbf{u}' - \mathbf{u}_0| + |\mathbf{u}_0 - \mathbf{u}''|. \tag{7.11}$$

First we give an upper bound to $|\mathbf{u}' - \mathbf{u}_0|$. Let $j = j(\tau) \leqq k$ be the smallest integer such that $\theta_j \geqq \tau - \pi/2$ and

$$2 > \sum_{i=j+1}^{k} l_i \cdot \sin(\tau - \theta_i).$$

Let l_j^* be the largest positive real number such that $l_j^* \leqq l_j$ and

$$2 \geqq l_j^* \cdot \sin(\tau - \theta_j) + \sum_{i=j+1}^{k} l_i \cdot \sin(\tau - \theta_i). \tag{7.12}$$

Simple geometric consideration shows that

$$|\mathbf{u}' - \mathbf{u}_0| \leqq l_j^* + \sum_{i=j+1}^{k} l_i. \tag{7.13}$$

We claim that

$$l_j^* + \sum_{i=j+1}^{k} l_i \leqq 21(l_{k-2} + l_{k-1}) + l_k \quad \text{provided} \quad j = j(\tau) \leqq k-4. \tag{7.14}$$

Using the elementary inequality $\sin x \geqq \frac{2}{\pi} x$ for $0 \leqq x \leqq \frac{\pi}{2}$, by (7.12) we get

$$2 \geqq l_j^* \cdot \frac{2}{\pi}(\tau - \theta_j) + \sum_{i=j+1}^{k} l_i \cdot \frac{2}{\pi}(\tau - \theta_i). \tag{7.15}$$

At least one of the numbers $k-1$ and $k-2$ is $\not\equiv 0 \pmod{m}$. Let (say) $k-2 \not\equiv 0 \pmod{m}$. Then by (7.7) we have for any $i \leqq k-3$,

$$\tau - \theta_i \geqq \theta_{k-1} - \theta_{k-3} \geqq \frac{\pi}{20 l_{k-2}}. \tag{7.16}$$

Combining (7.15) and (7.16) we have

$$2 \geqq \frac{2}{\pi} \cdot \frac{\pi}{20 l_{k-2}} \cdot \left(l_j^* + \sum_{i=j+1}^{k-3} l_i\right),$$

that is,

$$20 l_{k-2} \geqq l_j^* + \sum_{i=j+1}^{k-3} l_i. \tag{7.17}$$

Now by (7.17) we have

$$l_j^* + \sum_{i=j+1}^{k} l_i = l_j^* + \sum_{i=j+1}^{k-3} l_i + l_{k-2} + l_{k-1} + l_k \leqq 21(l_{k-2} + l_{k-1}) + l_k,$$

and (7.14) follows.

From (7.13) and (7.14) it follows by a little calculation that

$$|\mathbf{u}'-\mathbf{u}_0|^2 \leqq \Big(l_j^*+\sum_{i=j+1}^{k} l_i\Big)^2 \leqq (42)^2\cdot\Big((l_j^*)^2+\sum_{i=j+1}^{k} l_i^2\Big). \tag{7.18}$$

For every $i\in[1, m]$ and $\theta\in[\theta_i-\pi, \theta_i+\pi)$, let

$$g_i(\theta)=\begin{cases}\min\left\{l_i, \dfrac{\pi}{|\theta-\theta_i|}\right\}, & \text{if } \theta\in\left[\theta_i-\dfrac{\pi}{2}, \theta_i+\dfrac{\pi}{2}\right]\\ 0, \text{ if } \theta\in[\theta_i-\pi, \theta_i+\pi)\diagdown\left[\theta_i-\dfrac{\pi}{2}, \theta_i+\dfrac{\pi}{2}\right], \end{cases}$$

and extend it to a 2π-periodic function $g_i(x)$, $x\in\mathbf{R}$. From (7.15) immediately follows that $l_j^*\leqq\dfrac{\pi}{\tau-\theta_j}$ and $l_i\leqq\dfrac{\pi}{\tau-\theta_i}$ for all $j=j(\tau)<i\leqq k$. Thus by (7.18) we have

$$|\mathbf{u}'-\mathbf{u}_0|^2 \leqq (42)^2\cdot\Big((l_j^*)^2+\sum_{i=j+1}^{k} l_i^2\Big) \leqq (42)^2\cdot\Big(\sum_{i=1}^{m}(g_i(\tau))^2\Big). \tag{7.19}$$

Repeating the same argument, we obtain that

$$|\mathbf{u}_0-\mathbf{u}''|^2 \leqq (42)^2\cdot\Big(\sum_{i=1}^{m}(g_i(\tau))^2\Big). \tag{7.20}$$

Combining (7.10), (7.11), (7.19) and (7.20) we have

$$\big(f_B(\tau, d(\tau))\big)^2 \leqq |\mathbf{u}'-\mathbf{u}''|^2 \leqq 2(|\mathbf{u}'-\mathbf{u}_0|^2+|\mathbf{u}_0-\mathbf{u}''|^2) \leqq (84)^2\cdot\Big(\sum_{i=1}^{m}(g_i(\tau))^2\Big). \tag{7.21}$$

A little calculation shows that for any $i\in[1, m]$,

$$\int_0^{2\pi}(g_i(\tau))^2\,d\tau < 4\pi\cdot l_i. \tag{7.22}$$

Thus by (7.21) and (7.22) we conclude that

$$\int_0^{2\pi}\big(f_B(\tau, d(\tau))\big)^2\,d\tau \leqq (84)^2\cdot\sum_{i=1}^{m}\int_0^{2\pi}(g_i(\tau))^2\,d\tau < \tag{7.23}$$

$$< (84)^2\cdot 4\pi\Big(\sum_{i=1}^{m} l_i\Big) = 4\pi\cdot(84)^2\cdot l(Q).$$

We clearly have (note that $r(B)\geqq 2$) $l(Q)\leqq l(D)\leqq 4d(D)\leqq 4(d(B)+2)\leqq 6d(B)$. Hence by (7.23) we have

$$\int_0^{2\pi}\big(f_B(\tau, d(\tau))\big)^2\,d\tau < 4\pi\cdot(84)^2\cdot 6\cdot d(B) < 10^6\cdot d(B). \tag{7.24}$$

Finally, from (6.2), (6.4), (7.9) and (7.24) it follows that

$$\int_0^{2\pi} (f_B(\tau, 1))^2 \, d\tau \leqq 4 \int_0^{2\pi} (f_B(\tau, d(\tau)))^2 \, d\tau < 4 \cdot 10^6 \cdot d(B),$$

i.e., (7.1) holds with $c_{10} = 4 \cdot 10^6$.

Thus Lemma 4.2 is proved. ∎

This completes the proof of Theorem 1.1. ∎

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